Jour. Ind. Soc. Ag. Statistics
44 *(1), 1992 : 15-26*

Generalized Class of Estimators of Finite Population Mean Using Multi-auxiliary Information When Pairwise Correlation Coefficients are known

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(Received : December, 1985)

Summary

Srivastava and Jhajj [2] have defined a class of estimators of the population mean utilising information on population means as well as population variances of p (p > 1) auxiliary variables. In the present paper, the authors have extended this class to include information on pairwise correlation coefficients. The estimators of the class involve unknown constants whose optimum values depend upon unknown population parameters. It is shown that when these population parameters are replaced by their consistent estimators, the resulting estimators have the same asymptotic mean square error. An expression by which the mean square error of such estimators is smaller than those which use only population means and population variances of the auxiliary variables is also obtained.

Key words : Finite population, auxiliary information, asymptotic mean square error, efficiency.

Introduction

The problem of estimation of finite population mean or total of a main variable y using auxiliary information based on the variable x has been considered by many authors in survey sampling. Srivastava and Jhajj [1] defined a class of estimators for estimating the population mean $\bar{Y}$ by using information on auxiliary variable x in the form of known population mean and population variance of auxiliary variable. Srivastava and Jhajj [2] extended the class of estimators defined by them [1] to the case when information on population mean and population variance of more than one auxiliary variable is available.

Sometimes, the information of population correlation coefficients between each pair of the auxiliary variables is also available in addition to the knowledge of its population means and population variances. In such a situation, the class of estimators defined by Srivastava and Jhajj [2] is generalised in which information on population means, population variances and population correlation coefficients between each pair of the p (p>1) auxiliary variables is used. Asymptotic expression for the mean square error of any estimator of the class and also its minimum value is obtained. This minimum value is attained for optimum values of the constants in the estimator which depend on unknown population parameters. When these population parameters are replaced by their consistent estimators, the resulting estimators are shown to have the same asymptotic mean square error. An expression by which this asymptotic mean square is smaller than that of the estimators which use only the known population means and population variances of the auxiliary variable is also obtained.

2. *Notations and Results*

Let $x_1, x_2, \ldots, x_p$ be p variables having known pairwise correlation coefficients, means and variances. The values of the study variable y and auxiliary variable x_i ($i = 1, 2, \ldots, p$) on the j-th unit of the population are denoted by Y_j and X_{ij} ($j = 1, 2, \ldots, N$) respectively and the corresponding small letters denoted the values in the sample. Assume that a simple random sample of size n is drawn from the given finite population of size N. Write

$$\bar{y} = \frac{1}{n} \sum_{i=1}^{n} y_i \qquad \bar{Y} = \frac{1}{N} \sum_{i=1}^{N} Y_i$$

$$\bar{x}_i = \frac{1}{n} \sum_{j=1}^{n} x_{ij} \qquad \bar{X}_i = \frac{1}{N} \sum_{j=1}^{N} X_{ij}$$

$$s_0^2 = \frac{1}{n-1} \sum_{j=1}^{n} (y_j - \bar{y})^2 \qquad S_0^2 = \frac{1}{N-1} \sum_{j=1}^{N} (Y_j - \bar{Y})^2$$

$$s_i^2 = \frac{1}{n-1} \sum_{j=1}^{n} (x_{ij} - \bar{x}_i)^2 \qquad S_i^2 = \frac{1}{N-1} \sum_{j=1}^{N} (X_{ij} - \bar{X}_i)^2$$

$$s_{ik} = \frac{1}{n-1} \sum_{j=1}^{n} (x_{ij} - \bar{x}_i)(x_{kj} - \bar{x}_k) \ ; \ i \neq k$$

$$S_{ik} = \frac{1}{N-1} \sum_{j=1}^{N} (X_{ij} - \bar{X}_i)(X_{kj} - \bar{X}_k) \; ; \; i \neq k$$

$$s_{oi} = \frac{1}{n-1} \sum_{j=1}^{n} (y_j - \bar{y})(x_{ij} - \bar{x}_i)$$

$$S_{oi} = \frac{1}{N-1} \sum_{j=1}^{N} (Y_j - \bar{Y})(X_{ij} - \bar{X}_i)$$

$$r_{oi} = \frac{s_{oi}}{s_o \, s_i} \qquad \rho_{oi} = \frac{S_{oi}}{S_o \, S_i}$$

$$C_o = \frac{S_o}{\bar{Y}} \qquad C_i = \frac{S_i}{\bar{X}_i}$$

$$\mu_{\alpha\beta}(i) = \frac{1}{N} \sum_{m=1}^{N} (Y_m - \bar{Y})^{\alpha} (X_{im} - \bar{X}_i)^{\beta}$$

$$\mu_{\alpha\beta}(i, j) = \frac{1}{N} \sum_{m=1}^{N} (X_{im} - \bar{X}_i)^{\alpha} (X_{jm} - \bar{X}_j)^{\beta}$$

$$\mu_{\alpha\beta\gamma}(i, j) = \frac{1}{N} \sum_{m=1}^{N} (Y_m - \bar{Y})^{\alpha} (X_{im} - \bar{X}_i)^{\beta} (X_{jm} - \bar{X}_j)^{\gamma}$$

$$\mu_{\alpha\beta\gamma}(i, j, k) = \frac{1}{N} \sum_{m=1}^{N} (X_{im} - \bar{X}_i)^{\alpha} (X_{jm} - \bar{X}_j)^{\beta} (X_{km} - \bar{X}_k)^{\gamma}$$

$$\mu_{\alpha\beta\gamma\delta}(i, j, k, l) = \frac{1}{N} \sum_{m=1}^{N} (X_{im} - \bar{X}_i)^{\alpha} (X_{jm} - \bar{X}_j)^{\beta} (X_{km} - \bar{X}_k)^{\gamma} (X_{lm} - \bar{X}_l)^{\delta}$$

$$\lambda_{\alpha\beta}(i) = \frac{\mu_{\alpha\beta}(i)}{\mu_{20}^{\alpha/2}(i) \; \mu_{02}^{\beta/2}(i)}$$

$$\lambda_{\alpha\beta}(i, j) = \frac{\mu_{\alpha\beta}(i, j)}{\mu_{02}^{\alpha/2}(i) \; \mu_{02}^{\beta/2}(j)}$$

$$\lambda_{\alpha\beta\gamma}(i, j) = \frac{\mu_{\alpha\beta\gamma}(i, j)}{\mu_{200}^{\alpha/2}(i, j)\ \mu_{02}^{\beta/2}(i)\ \mu_{02}^{\gamma/2}(j)}$$

$$\lambda_{\alpha\beta\gamma}(i, j, k) = \frac{\mu_{\alpha\beta\gamma}(i, j, k)}{\mu_{02}^{\alpha/2}(i)\ \mu_{02}^{\beta/2}(j)\ \mu_{02}^{\gamma/2}(k)}$$

$$\lambda_{\alpha\beta\gamma\delta}(i, j, k, l) = \frac{\mu_{\alpha\beta\gamma\delta}(i, j, k, l)}{\mu^{\alpha/2}(i)\ \mu^{\beta/2}(j)\ \mu^{\gamma/2}(k)\ \mu^{\delta/2}(l)}$$

obviously,

$$\lambda_{11}(i) = \rho_{io} \quad \text{and} \quad \lambda_{11}(i, j) = \rho_{ij}$$

Though the following results could be obtained for any population size, for the sake of simplicity, assume that n is small as compared to N so that f.p.c. terms are ignored.

Define

$$u_i = \frac{\bar{x}_i}{\bar{X}_i} \qquad i = 1, 2, \ldots, p$$

$$v_i = \frac{s_i^2}{S_i^2} \qquad i = 1, 2, \ldots, p$$

and

$$w_i = \frac{s_{j,\,j+k}}{S_{j,\,j+k}}$$

$$i = \left\{(j-1)p+1-\sum_{l=0}^{j-1} l\right\}, \left\{(j-1)p+1-\sum_{l=0}^{j-1} l\right\}+1, \ldots, \left\{jp-\sum_{l=0}^{j} l\right\}$$

$$j = 1, 2, \ldots, p-1,$$

$$k = 1, 2, \ldots, p-j.$$

Let $\underline{u}$, $\underline{v}$ and $\underline{w}$ denote the column vectors formed by three sets of elements $u_1, u_2, \ldots, u_p$; $v_1, v_2, \ldots, v_p$ and $w_1, w_2, \ldots, w_{p'}$ respectively, and $\underline{z}$ denotes the column vector of p^* elements $u_1, u_2, \ldots, u_p$; $v_1, v_2, \ldots, v_p$ and $w_1, w_2, \ldots, w_{p'}$. Where $p^* = 2p + p'$ and $p' = \frac{p(p-1)}{2}$. Superfix T over a column vector denotes the corresponding row vector.

Defining

$$\delta_0 = \frac{\bar{y}}{\bar{Y}} - 1$$

$$\varepsilon_i = u_i - 1 \qquad i = 1, 2, \ldots, p$$

$$\delta_i = v_i - 1 \qquad i = 1, 2, \ldots, p$$

$$\eta_i = w_i - 1 \qquad i = 1, 2, \ldots, p'$$

and

$$\underline{\varepsilon}^T = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_p), \quad \underline{\delta}^T = (\delta_1, \delta_2, \ldots, \delta_p)$$

$$\underline{\eta}^T = (\eta_1, \eta_2, \ldots, \eta_{p'}), \ \underline{\theta}^T = (\underline{\varepsilon}^T : \underline{\delta}^T : \underline{\eta}^T)$$

We have

$$E(\delta_0) = 0, \ E(\varepsilon_i) = E(\delta_i) = 0 \qquad i = 1, 2, \ldots, p$$

$$E(\eta_i) = 0 \qquad i = 1, 2, \ldots, p'$$

$$E(\delta_0^2) = \frac{1}{n} C_0^2$$

$$E(\delta_0 \varepsilon_i) = \frac{1}{n} a_i \qquad i = 1, 2, \ldots, p$$

$$E(\delta_0 \delta_i) = \frac{1}{n} b_i \qquad i = 1, 2, \ldots, p$$

$$E(\delta_0 \eta_i) = \frac{1}{n} q_{i'} \qquad i' = 1, 2, \ldots, p'$$

$$E(\varepsilon_i \varepsilon_{i'}) = \frac{1}{n} e_{ii'} \qquad i, i' = 1, 2, \ldots, p$$

$$E(\varepsilon_i \delta_{i'}) = \frac{1}{n} f_{ii'} \qquad i, i' = 1, 2, \ldots, p$$

$$E(\varepsilon_i \eta_{i'}) = \frac{1}{n} m_{ii'} \qquad i = 1, 2, \ldots, p; \quad i' = 1, 2, \ldots, p'$$

and upto terms of order n^{-1}

$$E(\delta_i \delta_{i'}) = \frac{1}{n} h_{ii'} \qquad i, i' = 1, 2, \ldots, p$$

$$E(\delta_i \eta_{i'}) = \frac{1}{n} k_{ii'} \qquad i = 1, 2, \ldots, p; \quad i' = 1, 2, \ldots, p'$$

$$E(\eta_i \eta_{i'}) = \frac{1}{n} w_{ii'} \qquad i, i' = 1, 2, \ldots, p'$$

where

$$a_i = \rho_{oi} C_o C_i \qquad e_{ii'} = \rho_{ii'} C_i C_{i'}\ ; \ b_i = \lambda_{12}(i) C_o$$

$$f_{ii'} = \lambda_{12}(i, i') C_i \qquad h_{ii'} = \lambda_{22}(i, i') - 1$$

$$q_{i'} = \frac{\lambda_{111}(j, j+k) C_o}{\rho_{j, j+k}} \qquad m_{ii'} = \frac{\lambda_{111}(j, j, j+k)}{\rho_{j, j+k}}$$

$$k_{ii'} = \frac{\lambda_{211}(j, j, j+k)}{\rho_{j, j+k}} - 1$$

$$i = 1, 2, \ldots, p$$

$$i' = \left\{(j-1)p + 1 - \sum_{l=0}^{j-1} l\right\}, \left\{(j-1)p + 1 - \sum_{l=0}^{j-1} l\right\} + 1, \ldots, \left\{jp - \sum_{l=0}^{j} l\right\}$$

$$j = 1, 2, \ldots, p-1, \quad k = 1, 2, \ldots, p-j$$

$$w_{ii'} = \frac{\lambda_{1111}(j, j+k, j', j'+k')}{\rho_{j, j+k}\, \rho_{j', j'+k'}} - 1$$

$$i = \left\{(j-1)p + 1 - \sum_{l=0}^{j-1} l\right\}, \left\{(j-1)p - \sum_{l=0}^{j-1} l\right\} + 1, \ldots, \left\{jp - \sum_{l=0}^{j} l\right\}$$

$$j = 1, 2, \ldots, p-1, \qquad k = 1, 2, \ldots, p-j$$

$$i' = \left\{(j'-1)p + 1 + \sum_{l=0}^{j'-1} l\right\}, \left\{(j'-1)p + 1 - \sum_{l=0}^{j'-1} l\right\} + 1, \ldots, \left\{j'p - \sum_{l=0}^{j'} l\right\}$$

$$j' = 1, 2, \ldots, p-1, \qquad k' = 1, 2, \ldots, p-j'$$

In matrix notations, we have

$$E(\underline{\varepsilon}) = E(\underline{\delta}) = E(\underline{\eta}) = \underline{O}$$

$$E(\delta_o \underline{\varepsilon}) = \frac{1}{n}\underline{a}, \quad E(\delta_o \underline{\delta}) = \frac{1}{n}\underline{b}, \quad E(\delta_o \underline{\eta}) = \frac{1}{n}\underline{q}$$

$$E(\underline{\varepsilon}\,\underline{\varepsilon}^T) = \frac{1}{n}\underline{E}, \quad E(\underline{\varepsilon}\,\underline{\delta}^T) = \frac{1}{n}\underline{F}, \quad E(\underline{\varepsilon}\,\underline{\eta}^T) = \frac{1}{n}\underline{M}$$

$$E(\underline{\delta}\,\underline{\delta}^T) = \frac{1}{n}\underline{H}, \quad E(\underline{\delta}\,\underline{\eta}^T) = \frac{1}{n}\underline{K}, \quad E(\underline{\eta}\,\underline{\eta}^T) = \frac{1}{n}\underline{W}$$

where the vectors

$$\underline{a}^T = (a_1, a_2, \ldots, a_p), \quad \underline{b}^T = (b_1, b_2, \ldots, b_p), \quad \underline{q}^T = (q_1, q_2, \ldots, q_{p'})$$

and the matrices

$$\underline{E} = [e_{ii'}]_{pxp}, \quad \underline{F} = [f_{ii'}]_{pxp}$$

$$\underline{H} = [h_{ii'}]_{pxp}, \quad \underline{M} = [m_{ii'}]_{pxp'}$$

$$\underline{K} = [k_{ii'}]_{pxp'}, \quad \underline{W} = [w_{ii'}]_{p'xp'}$$

Above results may be written as

$$E(\delta_o \underline{\theta}) = \frac{1}{n}\underline{d}, \quad E(\underline{\theta}\,\underline{\theta}^T) = \frac{1}{n}\underline{D}$$

where the vector $\underline{d}^T = (\underline{a}^T : \underline{b}^T : \underline{q}^T)$

and the matrix

$$\underline{D} = \begin{pmatrix} \underline{E} & \underline{F} & \underline{M} \\ \underline{F}^T & \underline{H} & \underline{K} \\ \underline{M}^T & \underline{K}^T & \underline{W} \end{pmatrix}$$

which is assumed to be positive definite.

3. *The class of estimators*

In this section, the class of estimators defined by Srivastava and Jhajj [2] is extended to the case in which information on pairwise correlation coefficients ρ_{ij} ($1 < j = 1, 2, \ldots, p$) is also used. Since $S_{ij} = \rho_{ij} S_i S_j$, if ρ_{ij}, S_i and S_j are known then S_{ij} is known.

Whatever the sample chosen, let $\underline{z}$ be assuming values in a closed convex subset R, of the p^* dimensional real space containing the point $e = \underline{e} = \begin{pmatrix} \underline{e}_1 \\ \underline{e}_2 \\ \underline{e}_3 \end{pmatrix}$, the column vector of p^* unit elements. Let $g(\underline{u}, \underline{v}, \underline{w})$ be a function of $(\underline{u}, \underline{v}, \underline{w})$ such that

$$g(\underline{e}_1, \underline{e}_2, \underline{e}_3) = g(\underline{e}) = 1 \tag{3.1}$$

and which is continuous and bounded having continuous and bounded first and second order partial derivatives in R.

The class of estimators is defined by

$$\tilde{y}_g = \bar{y}\, g(\underline{u}, \underline{v}, \underline{w}) = \bar{y}\, g(\underline{z}) \tag{3.2}$$

To find the bias and mean square error of $\tilde{y}_g$, expand the function about the point $\underline{e}$ in the second order Taylor's series, and obtain

$$\tilde{y}_g = \bar{y}\Bigg[g(\underline{e}_1, \underline{e}_2, \underline{e}_3) + \left\{(\underline{u}-\underline{e}_1)^T (\underline{v}-\underline{e}_2)^T (\underline{w}-\underline{e}_3)^T\right\} \begin{pmatrix} g^{(1)}(\underline{e}) \\ g^{(2)}(\underline{e}) \\ g^{(3)}(\underline{e}) \end{pmatrix}$$

$$+ \frac{1}{2} - \left\{(\underline{u}-\underline{e}_1)^T (\underline{v}-\underline{e}_2)^T (\underline{w}-\underline{e}_3)^T\right\} \begin{pmatrix} g^{(11)}(\underline{z}^*) & \vdots & g^{(12)}(\underline{z}^*) & \vdots & g^{(13)}(\underline{z}^*) \\ \cdots & & \cdots & & \cdots \\ g^{(21)}(\underline{z}^*) & \vdots & g^{(22)}(\underline{z}^*) & \vdots & g^{(23)}(\underline{z}^*) \\ \cdots & & \cdots & & \cdots \\ g^{(31)}(\underline{z}^*) & \vdots & g^{(32)}(\underline{z}^*) & \vdots & g^{(33)}(\underline{z}^*) \end{pmatrix}$$

$$\begin{pmatrix} \underline{u} - \underline{e}_1 \\ \underline{v} - \underline{e}_2 \\ \underline{w} - \underline{e}_3 \end{pmatrix}\Bigg]$$

$$= \bar{y}\left[g(\underline{e}) + (\underline{z}-\underline{e})^T g'(\underline{e}) + \frac{1}{2}(\underline{z}-\underline{e})^T g''(\underline{z}^*)(\underline{z}-\underline{e}) \right] \tag{3.3}$$

where $\underline{z}^* = \underline{e} + \theta_1(\underline{z}-\underline{e})$, $0 < \theta_1 < 1$ and $g'(\underline{e})$ denotes the column vector of first order partial derivatives of $g(\underline{z})$ at the point $\underline{z} = \underline{e}$; $g''(\underline{z}^*)$ denotes the matrix of second order partial derivatives of $g(\underline{z})$ with respect to $u_1, u_2, \ldots, u_p, v_1, v_2, \ldots, v_p, w_1, w_2, \ldots, w_p,$ at the point $\underline{z} = \underline{z}^*$. Substituting $\bar{y}$ and $\underline{z}$ in terms of δ_o and $\underline{\theta}$ in (3.3), we have

$$\tilde{y}_g = \bar{Y}(1+\delta_o)\left[1+\underline{\theta}^T g'(\underline{e})+\frac{1}{2}\underline{\theta}^T g''(\underline{z}^*)\underline{\theta}\right]$$

$$= \bar{Y}\left[1+\delta_o+\underline{\theta}^T g'(\underline{e})+\underline{\delta}\,\underline{\theta}^T g'(\underline{e})+\frac{1}{2}\underline{\theta}^T g''(\underline{z}^*)\underline{\theta}+\frac{1}{2}\delta_o\underline{\theta}^T g''(\underline{z}^*)\underline{\theta}\right] \quad (3.4)$$

Taking expectation and noting that $g''(\underline{z}^*)$ is bounded it is seen that bias of $\tilde{y}_g$ is of the order of n^{-1}. Hence its contribution to mean square error will be of order of n^{-2}. The mean square error of $\tilde{y}_g$ is given by

$$MSE(\tilde{y}_g) = E[\tilde{y}_g - \bar{Y}]^2$$

$$= \bar{Y}^2 E\left[\delta_o^2 + 2\delta_o\underline{\theta}^T g'(\underline{e})+\{g'(\underline{e})\}^T \underline{\theta}\,\underline{\theta}^T g'(e)\right]$$

$$= \frac{1}{n}\bar{Y}^2\left[C_y^2+2\underline{d}^T g'(\underline{e})+\{g'(\underline{e})\}^T \underline{D}\,g'(\underline{e})\right] \quad (3.5)$$

The mean square error (3.5) is minimised for

$$g'(\underline{e}) = -\underline{D}^{-1}\underline{d} \quad (3.6)$$

where

$$\underline{D}^{-1} = \begin{pmatrix} E^{-1}(I+FB^{-1}F^TE^{-1})+\underline{V}\underline{Q}^{-1}\underline{V}^T & \vdots & -\underline{E}^{-1}\,\underline{FB}^{-1}-\underline{VQ}^{-1}(\underline{B}^{-1}\underline{R})^T] & \vdots & \underline{-VQ}^{-1} \\ \cdots & & \cdots & & \cdots \\ -[\underline{E}^{-1}\,\underline{FB}^{-1}-\underline{VQ}^{-1}(\underline{B}^{-1}\underline{R})^T]^T & \vdots & \underline{B}^{-1}[(I+\underline{RQ}^{-1}(\underline{BR})^T] & \vdots & -\underline{B}^{-1}\,\underline{RQ}^{-1} \\ \cdots & & \cdots & & \cdots \\ -(\underline{VQ}^{-1})^T & \vdots & -(\underline{B}^{-1}\underline{RQ}^{-1})^T & \vdots & \underline{Q}^{-1} \end{pmatrix}$$

and

$$\underline{B} = \underline{H} - \underline{F}^T\underline{E}^{-1}\underline{F} \qquad \underline{V} = \underline{E}^{-1}(\underline{M} - \underline{F}\,\underline{B}^{-1}\underline{R})$$

$$\underline{R} = \underline{K} - \underline{F}^T\underline{E}^{-1}\underline{M} \qquad \underline{Q} = \underline{G} - \underline{R}^T\underline{B}^{-1}\underline{R}$$

$$\underline{G} = \underline{W} - \underline{M}^T\underline{E}^{-1}\underline{M}$$

The minimum value of mean square error of $\tilde{y}_g$ upto terms of order n^{-1} is given by

$$\min MSE(\tilde{y}_g) = \frac{\bar{Y}^2}{n}\left[C_y^2 - \underline{d}^T \underline{D}^{-1} \underline{d}\right]$$

$$= \frac{\bar{Y}^2}{n} [C_y^2 - \underline{a}^T\underline{E}^{-1}\underline{a} - \{\underline{F}^T\underline{E}^{-1}\underline{a} - \underline{b}\}^T \underline{B}^{-1}\{\underline{F}^T\underline{E}^{-1}\underline{a} - \underline{b}\}$$

$$- \{\underline{V}'\underline{a} + (\underline{B}^{-1}\underline{R})^T\underline{b} - \underline{q}\}^T \underline{Q}^{-1} \{\underline{V}'\underline{a} + (\underline{B}^{-1}\underline{R})^T\underline{b} - \underline{q}\}]$$

$$= \frac{\bar{Y}^2}{n} [C_y^2(1-R^2) - \{\underline{F}^T\underline{E}^{-1}\underline{a} - \underline{b}\}^T \underline{B}^{-1}\{\underline{F}^T\underline{E}^{-1}\underline{a} - \underline{b}\}$$

$$- \{\underline{V}'\underline{a} + (\underline{B}^{-1}\underline{R})^T\underline{b} - \underline{q}\}^T \underline{Q}^{-1} \{\underline{V}'\underline{a} + (\underline{B}^{-1}\underline{R})^T\underline{b} - \underline{q}\}] \quad (3.7)$$

where R stands for multiple correlation coefficient of y on $x_1, x_2, \ldots, x_p$. Since B and Q are positive definite the second and third term on the right had side of (3.7) are non negative.

On the right hand side of (3.7), the first term is the asymptotic mean square error of the multiple linear regression estimator and the second term gives amount of reduction in the asymptotic mean square error when knowledge of the population variances of the auxiliary variables along with their means is only used. Hence third term on the right hand side of (3.7) gives the amount of reduction in the asymptotic mean square error when the knowledge of the pairwise population correlation coefficients with their means and variances of the auxiliary variables is also used.

The class of estimators (3.2) is very large. Any parametric function g(z) such that g(e) = 1 and satisfying the conditions given earlier can generate an estimator of the class $\tilde{y}_g$.

The following functions, for example, give some simple estimators of the class.

(i) $g(\underline{z}) = 1 + \underline{\alpha}^T (\underline{z} - \underline{e})$

(ii) $g(\underline{z}) = \exp\{\underline{\alpha}^T (\underline{z} - \underline{e})\}$

(iii) $g(\underline{z}) = \dfrac{1}{1 + \underline{\alpha}^T (\underline{z} - \underline{e})}$

where $\underline{\alpha}^T = (\alpha_1, \alpha_2, \ldots, \alpha_{p^*})$ is a vector of p constants. The optimum values of these constants are determined from conditions (3.6). Since (3.6) contains p^* equations, we have taken exactly p^* constants in defining estimators of the class. We assume that equations(3.6) can be solved uniquely for the p^* unknown constants in the class of estimators (3.2). The optimum values of these constants will involve which is function of unknown parameters. It is understood that when these optimum values are substituted in (3.2) it no longer remains an estimator since it involves unknown $\underline{\varphi} = \underline{D}^{-1}\underline{d}$. Let $\hat{\underline{\varphi}}$ be a consistent estimator of $\underline{\varphi}$. If necessary in the optimum $\tilde{y}_g$, we replace $\underline{\varphi}$ by $\hat{\underline{\varphi}}$ resulting an estimator $\tilde{y}^*_g$.

We write

$$\tilde{y}^*_g = \bar{y}\, g^*(\underline{z}, \hat{\underline{\varphi}}) \tag{3.8}$$

where the function $g^*(.,.)$ is obtained from the function g(.) given at (3.2) by replacing the unknown constants in it by the consistent estimators of their optimum values. Then the condition (3.1) will imply that

$$g^*(\underline{e}, \underline{\varphi}) = 1 \tag{3.9}$$

We further assume that

$$\left.\frac{\partial g^*}{\partial \underline{z}}\right|_{(\underline{e},\,\underline{\varphi})} = \left.\frac{\partial g}{\partial \underline{z}}\right|_{(\underline{e})} = -\,\underline{\varphi} \tag{3.10}$$

and

$$\left.\frac{\partial g^*}{\partial \underline{\varphi}}\right|_{(\underline{e},\,\underline{\varphi})} = 0 \tag{3.11}$$

Expanding $g^*(\underline{z}, \hat{\underline{\varphi}})$ about the point $(\underline{e}, \underline{\varphi})$ in a Taylor's series as in section III and using (3.9) to (3.11), we have

$$\tilde{y}^*_g = \bar{y}\, [g^*(\underline{e},\underline{\varphi}) + (\underline{z}-\underline{e})^T \left.\frac{\partial g^*}{\partial \underline{z}}\right|_{(\underline{e},\,\underline{\varphi})} + (\hat{\underline{\varphi}} - \underline{\varphi})^T \left.\frac{\partial g^*}{\partial \underline{\varphi}}\right|_{(\underline{e},\underline{\varphi})}$$

+ second order terms

$$= \bar{Y}(1+\delta_o)\left[1+\underline{\theta}^T \frac{\partial g^*}{\partial \underline{z}}\bigg|_{(\underline{e}.\underline{\varphi})} + (\hat{\underline{\varphi}}-\underline{\varphi})^T \frac{\partial g^*}{\partial \underline{\varphi}}\bigg|_{(\underline{e}.\underline{\varphi})} + \text{second order terms}\right]$$

$$= \bar{Y}\left[1+\delta_o + \underline{\theta}^T(-\underline{\varphi}) + \text{ second order terms}\right] \quad (3.12)$$

Taking expectation and noting that $\hat{\underline{\varphi}}$ is a consistent estimator of $\underline{\varphi}$. The bias of $\tilde{y}_g^*$ will be of order n^{-1}. Upto terms of order n^{-1}, the mean square error of $\tilde{y}_g^*$ from (3.12) is

$$\text{MSE}(\tilde{y}_g^*) = E\left[\tilde{y}_g^* - \bar{Y}\right]^2$$

$$= \frac{\bar{Y}^2}{n} E\left[\delta_o - \underline{\theta}^T \underline{\varphi}\right]^2$$

$$= \frac{\bar{Y}^2}{n}\left[C_y^2 - \underline{d}^T \underline{D}^{-1} \underline{d}\right]$$

$$= \frac{\bar{Y}^2}{n}[C_y^2(1-R^2)\{\underline{F}^T\underline{E}^{-1}\underline{a}-\underline{b}\}^T \underline{B}^{-1}\{\underline{F}^T\underline{E}^{-1}\underline{a}-\underline{b}\}$$
$$- \{\underline{V}'\underline{a} + (\underline{B}^{-1}\underline{R})^T \underline{b}-\underline{q}\}^T \underline{Q}^{-1}\{\underline{V}'\underline{a}+(\underline{B}^{-1}\underline{R})^T\underline{b}-\underline{q}\}]$$

which is same as (3.7). Hence it is concluded that if the optimum values of constants in (3.1) are replaced by their consistent estimators and conditions (3.10) and (3.11) hold, the resulting estimator $\tilde{y}_g^*$ has the same mean square error upto terms of order n^{-1}, as that of optimum $\tilde{y}_g$.

ACKNOWLEDGEMENT

The authors are thankful to Dr. S.K. Srivastava for his help during the preparation of this paper and also grateful to the referee for his suggestions leading to an improvement of the paper.

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